

Total nonnegativity and stable polynomials

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Abstract

We consider homogeneous multiaffine polynomials whose coefficients are the Plücker coordinates of a point V of the Grassmannian. We show that such a polynomial is stable (with respect to the upper half plane) if and only if V is in the totally nonnegative part of the Grassmannian. To prove this, we consider an action of matrices on multiaffine polynomials. We show that a matrix A preserves stability of polynomials if and only if A is totally nonnegative. The proofs are applications of classical theory of totally nonnegative matrices, and the generalized Pólya–Schur theory of Borcea and Brändén.

1 Introduction

A multivariate polynomial $f(\mathbf{x}) = f(x_1, \dots, x_n) \in \mathbb{C}[\mathbf{x}]$ is said to be *stable* if either $f \equiv 0$, or $f(\mathbf{u}) \neq 0$, for all $\mathbf{u} = (u_1, \dots, u_n) \in \mathcal{H}^n$, where $\mathcal{H} = \{u \in \mathbb{C} \mid \operatorname{Im}(u) > 0\}$ denotes the upper half plane in \mathbb{C} . The theory of stable polynomials generalizes and vastly extends the theory of univariate real polynomials with only real roots. Although the idea of considering polynomials (and more generally analytic functions) with no zeros inside a domain has an extensive history in complex analysis, more recent developments — notably the generalized Pólya–Schur theory of Borcea and Brändén [1, 2] — have generated new interest in the subject, and a wide variety of new applications have been discovered in areas such as matrix theory, statistical mechanics, and combinatorics. We refer the reader to the survey [18] for an introduction to the theory of stable polynomials and an overview of some of its applications.

Central to the theory is the vector space $\mathbb{C}^{\text{MA}}[\mathbf{x}]$ of *multiaffine* polynomials. These are the polynomials in $\mathbb{C}[\mathbf{x}]$ that have degree at most one in each individual variable. The Grace–Walsh–Szegő coincidence theorem [7, 17, 19] allows one to reduce many problems about stable polynomials to the multiaffine case; moreover, a number of applications of the theory, notably those involving matroid theory [4, 5], statistical mechanics [3], and the present paper, involve only multiaffine polynomials.

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The *Grassmannian* $\text{Gr}(k, n)$ is the space of all k -dimensional linear subspaces of \mathbb{C}^n . There are two common ways to specify a point $V \in \text{Gr}(k, n)$. The simplest is as the column space of a rank k complex matrix $M \in \text{Mat}(n \times k)$; however, for any given V , this matrix M is not unique. A more canonical way to specify V is via its Plücker coordinates. Let $M[I]$ denote the $k \times k$ submatrix of M with row set $I \in \binom{[n]}{k}$. The *Plücker coordinates* of V are the maximal minors $[\det(M[I]) : I \in \binom{[n]}{k}]$. These are homogeneous coordinates for V , i.e. they are well-defined up to rescaling by a nonzero constant. We can encode the Plücker coordinates of V into a homogeneous multiaffine polynomial of degree k : we will say that the polynomial

$$\sum_{I \in \binom{[n]}{k}} \det(M[I]) \mathbf{x}^I \in \mathbb{C}^{\text{MA}}[\mathbf{x}]$$

represents V , where $\mathbf{x}^I := \prod_{i \in I} x_i$. Not every homogeneous multiaffine polynomial of degree k represents a point of $\text{Gr}(k, n)$. A necessary and sufficient condition is that the coefficients satisfy the quadratic Plücker relations, the defining equations for $\text{Gr}(k, n)$ as a projective variety.

If $V \in \text{Gr}(k, n)$ is the column space of a matrix M whose maximal minors are all nonnegative, we say that V is *totally nonnegative*. The *totally nonnegative part* of the Grassmannian, denoted $\text{Gr}_{\geq 0}(k, n)$, is the set of all totally nonnegative $V \in \text{Gr}(k, n)$.

The totally nonnegative part of a flag variety (the Grassmannian being the most important example) was first introduced by Lusztig [13], as a part of a generalization of the classical theory of totally nonnegative matrices. Rietsch showed that totally nonnegative part of any flag variety has a decomposition into cells [16]; Marsh and Rietsch described a parameterization of the cells [14]. In the case of the Grassmannian, Lusztig's definition agrees with the definition above. Postnikov described the indexing of the cells $\text{Gr}_{\geq 0}(k, n)$ and their parameterizations in combinatorially explicit ways [15], making $\text{Gr}_{\geq 0}(k, n)$ a very accessible object. Total nonnegativity has played a key role in a number of recent applications. Some of these include: the development of cluster algebras [6]; soliton solutions to the KP equation [11]; the (remarkably well-behaved) positroid stratification of the Grassmannian [10], which has applications to geometric Schubert calculus [9]. Our first main result relates total nonnegativity on the Grassmannian to stable polynomials.

Theorem 1.1. *Suppose $f(\mathbf{x}) \in \mathbb{C}^{\text{MA}}[\mathbf{x}]$ is a homogeneous multiaffine polynomial of degree k that represents a point $V \in \text{Gr}(k, n)$. Then $f(\mathbf{x})$ is stable if and only if V is totally nonnegative.*

The “phase theorem” of Choe, Oxley, Sokal, and Wagner [5, Theorem 6.1] asserts that if $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ is stable and homogeneous, then all of its coefficients have the same complex phase, i.e. there is a scalar $\alpha \in \mathbb{C}^\times$ such that all terms of $\alpha f(\mathbf{x})$ have nonnegative real coefficients. The “only if” direction of Theorem 1.1 is an immediate consequence. In general, however, the converse of the phase theorem is false: for example, $x_1 x_2 + x_3 x_4$ is not stable. Although there are necessary and sufficient criteria

for a polynomial to be stable (see Theorem 2.1), they can be cumbersome to use in practice, and they do not readily yield an explicit description of the set of stable polynomials as a semialgebraic set. It is therefore interesting and surprising that adding a well-known algebraic condition on the coefficients (the Plücker relations) reduces the problem of testing stability to a simple nonnegativity condition. This can be seen quite explicitly in the case $k = 2$, $n = 4$; here, the necessary and sufficient conditions for stability are tractable, and the Plücker relation trivializes them (see Remark 2.3).

A point $V \in \text{Gr}(k, n)$ determines a representable matroid of rank k on the set $[n]$, by taking the bases to be the indices of the nonzero Plücker coordinates. If V is totally nonnegative, this matroid is called a *positroid*. The class of positroids is combinatorially well-behaved compared to the class of representable matroids. For example, positroids can be enumerated [21]. Recently, Marcott showed that positroids have the *Rayleigh property* [8], a property of matroids closely related to theory stable polynomials. This result indicates another relationship between $\text{Gr}_{\geq 0}(n, k)$ and stable polynomials; it has a similar flavour to Theorem 1.1, but neither theorem implies the other.

To prove Theorem 1.1, we establish a second connection between the theory of stable polynomials and total nonnegativity. Recall that a matrix $A \in \text{Mat}(n \times n)$ is *totally nonnegative* if all minors of A are nonnegative.

Let $\Lambda[\mathbf{x}]$ denote the complex exterior algebra generated by \mathbf{x} , with multiplication denoted \wedge , and relations $x_i \wedge x_j + x_j \wedge x_i = 0$, for $i, j \in [n]$. If $I = \{i_1 < i_2 < \dots < i_k\} \subset [n]$, write $\mathbf{x}^I := x_{i_1} \wedge x_{i_2} \wedge \dots \wedge x_{i_k}$. There is a unique vector space isomorphism $\xi : \mathbb{C}^{\text{MA}}[\mathbf{x}] \rightarrow \Lambda[\mathbf{x}]$ such that $\xi(\mathbf{x}^I) = \mathbf{x}^I$. Since $\Lambda[\mathbf{x}]$ is a $\text{Mat}(n \times n)$ -algebra, this isomorphism gives us a linear action of $\text{Mat}(n \times n)$ on $\mathbb{C}^{\text{MA}}[\mathbf{x}]$. Specifically, for $A \in \text{Mat}(n \times n)$, we have a linear endomorphism $A_{\#} : \mathbb{C}^{\text{MA}}[\mathbf{x}] \rightarrow \mathbb{C}^{\text{MA}}[\mathbf{x}]$,

$$A_{\#}f(\mathbf{x}) := \xi^{-1}(A\xi(f(\mathbf{x}))),$$

where $Ax_j := \sum_{i \geq 0} A_{ij}x_j$, and $A(x_{j_1} \wedge \dots \wedge x_{j_k}) := Ax_{j_1} \wedge \dots \wedge Ax_{j_k}$. An example of this construction is given in (3) as part of the proof of Lemma 2.6.

At first glance, the definition of $A_{\#}$ seems absurd: we have made a linear identification between part of a commutative algebra and a supercommutative algebra. In fact, this issue was already present when we took Plücker coordinates as coefficients of a polynomial. The intuition here is that the difference between these two structures is in the signs; when we restrict our attention to totally nonnegative matrices, or the totally nonnegative part of the Grassmannian, the signs are all positive, and the two structures become compatible.

Theorem 1.2. *For $A \in \text{Mat}(n \times n)$, the following are equivalent:*

- (a) *A is totally nonnegative;*
- (b) *for every stable polynomial $f(\mathbf{x}) \in \mathbb{C}^{\text{MA}}[\mathbf{x}]$, $A_{\#}f(\mathbf{x})$ is stable.*

In Section 2, we recall some of the major results from the theory of stable polynomials. We then apply this theory to obtain a key lemma, which is roughly the $n = 2$

case of Theorem 1.2. In Section 3, we discuss some pertinent elements of the theory of total nonnegativity and total positivity, for matrices and for the Grassmannian. We use these, and our results from Section 2 to prove Theorems 1.1 and 1.2. In Section 4 we look at a handful of related results, including other families of homogeneous multiaffine stable polynomials, a family of infinitesimal stability preservers, and a slightly stronger version of the phase theorem.

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2 Multiaffine stable polynomials

We begin with an example, in which we determine necessary and sufficient conditions for a degree 2 homogeneous polynomial in 4 variables to be stable. This turns out to be the fundamental brute-force calculation needed to prove our main theorems. To obtain such conditions, we use the following criterion for stability of multiaffine polynomials with real coefficients.

Theorem 2.1 (Brändén [4]). *If $f(\mathbf{x}) \in \mathbb{R}^{\text{MA}}[\mathbf{x}]$ is a multiaffine polynomial with real coefficients, define*

$$\Delta_{ij}f(\mathbf{x}) := \frac{\partial}{\partial x_i}f(\mathbf{x}) \cdot \frac{\partial}{\partial x_j}f(\mathbf{x}) - f(\mathbf{x}) \cdot \frac{\partial^2}{\partial x_i \partial x_j}f(\mathbf{x}).$$

Then $f(\mathbf{x})$ is stable if and only if $\Delta_{ij}f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a nonnegative function for all $i, j \in [n]$, $i \neq j$.

Example 2.2. Let $a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34} \geq 0$. Consider the polynomial

$$f(\mathbf{x}) = a_{12}x_1x_2 + a_{13}x_1x_3 + a_{14}x_1x_4 + a_{23}x_2x_3 + a_{24}x_2x_4 + a_{34}x_3x_4.$$

By Theorem 2.1, $f(\mathbf{x})$ is stable iff $\Delta_{ij}f \geq 0$ for all i, j . We compute

$$\Delta_{13}f(\mathbf{x}) = a_{12}a_{23}x_2^2 + (a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})x_2x_4 + a_{14}a_{34}x_4^2. \quad (1)$$

Since $a_{ij} \geq 0$, $\Delta_{13}f$ is nonnegative if and only if its discriminant is nonpositive, i.e.

$$a_{12}^2a_{34}^2 + a_{13}^2a_{24}^2 + a_{14}^2a_{23}^2 - 2a_{12}a_{34}a_{13}a_{24} - 2a_{13}a_{24}a_{14}a_{23} - 2a_{12}a_{34}a_{14}a_{23} \leq 0. \quad (2)$$

Since this expression is invariant under permutations of [4], we obtain the same inequality for every other pair of indices $i, j \in [4]$. Hence the inequality (2) is a necessary and sufficient condition for $f(\mathbf{x})$ to be stable.

Remark 2.3. $\text{Gr}(2, 4)$ is defined by a single Plücker relation: $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$. If this holds, then (1) is clearly nonnegative, and so (2) holds. This proves the $\text{Gr}(2, 4)$ case of Theorem 1.1. However, in general, it is not straightforward to deduce Theorem 1.1 from Theorem 2.1 using the Plücker relations.

A \mathbb{C} -linear map satisfying condition (b) of Theorem 1.2 is called a *stability preserver*. As part of their vast generalization of the Pólya–Schur theorem, Borcea and Brändén proved that there is an equivalence between stability preservers, and stable polynomials in twice as many variables. We state only the multiaffine case of their theorem, as we will not need the result in its full generality.

Theorem 2.4 (Borcea–Brändén [1]). *Let $\phi : \mathbb{C}^{\text{MA}}[\mathbf{x}] \rightarrow \mathbb{C}^{\text{MA}}[\mathbf{x}]$ be a \mathbb{C} -linear map. Then ϕ is a stability preserver if and only if one the following is true:*

- (a) *there is a linear functional $\eta : \mathbb{C}^{\text{MA}}[\mathbf{x}] \rightarrow \mathbb{C}$ and a stable polynomial $g(\mathbf{x}) \in \mathbb{C}^{\text{MA}}[\mathbf{x}]$ such that $\phi f(\mathbf{x}) = \eta(f(\mathbf{x}))g(\mathbf{x})$; or*
- (b) *$\phi(\prod_{i=1}^n (x_i + y_i)) \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is stable.*

We will refer to stability preservers satisfying (a) as *rank-one stability preservers*, and those satisfying (b) as *true stability preservers*. In (b), we are implicitly extending ϕ from a \mathbb{C} -linear map $\mathbb{C}^{\text{MA}}[\mathbf{x}] \rightarrow \mathbb{C}^{\text{MA}}[\mathbf{x}]$ to the unique $\mathbb{C}[\mathbf{y}]$ -linear map $\phi : \mathbb{C}^{\text{MA}}[\mathbf{x}, \mathbf{y}] \rightarrow \mathbb{C}^{\text{MA}}[\mathbf{x}, \mathbf{y}]$ that agrees with the original ϕ on $\mathbb{C}[\mathbf{x}]$. An important property of true stability preservers is that they are preserved by this natural type of extension.

Proposition 2.5. *A linear map $\phi : \mathbb{C}^{\text{MA}}[\mathbf{x}] \rightarrow \mathbb{C}^{\text{MA}}[\mathbf{x}]$ is a true stability preserver if and only if for any additional set of variables $\mathbf{z} = (z_1, \dots, z_m)$, the $\mathbb{C}[\mathbf{z}]$ -linear extension $\phi : \mathbb{C}^{\text{MA}}[\mathbf{x}, \mathbf{z}] \rightarrow \mathbb{C}^{\text{MA}}[\mathbf{x}, \mathbf{z}]$ is a true stability preserver.*

Proof. By definition, $\phi : \mathbb{C}^{\text{MA}}[\mathbf{x}] \rightarrow \mathbb{C}^{\text{MA}}[\mathbf{x}]$ is a true stability preserver iff

$$h(\mathbf{x}, \mathbf{y}) = \phi\left(\prod_{i=1}^n (x_i + y_i)\right)$$

is stable. The extension $\phi : \mathbb{C}^{\text{MA}}[\mathbf{x}, \mathbf{z}] \rightarrow \mathbb{C}^{\text{MA}}[\mathbf{x}, \mathbf{z}]$ is a true stability preserver iff

$$\phi\left(\prod_{i=1}^n (x_i + y_i) \cdot \prod_{j=1}^m (z_j + w_j)\right) = h(\mathbf{x}, \mathbf{y}) \prod_{j=1}^m (z_j + w_j)$$

is stable. It is straightforward to verify that $h(\mathbf{x}, \mathbf{y}) \in \mathbb{C}[\mathbf{x}, \mathbf{y}]$ is stable if and only if $h(\mathbf{x}, \mathbf{y}) \prod_{j=1}^m (z_j + w_j) \in \mathbb{C}[\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w}]$ is stable. The result follows. \square

In general, rank-one stability preservers do not have this extendability property, unless they are also true stability preservers.

The set of stable polynomials in $\mathbb{C}^{\text{MA}}[\mathbf{x}]$ is closed. It follows that the set of true stability preservers $\mathbb{C}^{\text{MA}}[\mathbf{x}] \rightarrow \mathbb{C}^{\text{MA}}[\mathbf{x}]$, being linearly equivalent to the set of stable polynomials in $\mathbb{C}^{\text{MA}}[\mathbf{x}, \mathbf{y}]$, is also closed. These facts will be used in the next section.

We conclude this section by using Theorem 2.4 to prove the following lemma, which is almost-but-not-quite the $n = 2$ case of Theorem 1.2.

Lemma 2.6. *If $Q \in \text{Mat}(2 \times 2)$ is totally nonnegative, then $Q_{\#} : \mathbb{C}^{\text{MA}}[x_1, x_2] \rightarrow \mathbb{C}^{\text{MA}}[x_1, x_2]$ is a true stability preserver.*

Proof. Write $Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. Then we have

$$\begin{aligned} Q_{\#}(1) &= 1 \\ Q_{\#}(x_1) &= ax_1 + bx_2 \\ Q_{\#}(x_2) &= cx_1 + dx_2 \\ Q_{\#}(x_1x_2) &= (ad - bc)x_1x_2. \end{aligned} \tag{3}$$

Thus $Q_{\#}$ is a true stability preserver if and only if

$$h(\mathbf{x}, \mathbf{y}) = y_1y_2 + ax_1y_2 + bx_2y_2 + cx_1y_1 + dx_2y_1 + (ad - bc)x_1x_2$$

is stable.

Now assume Q is totally nonnegative. Then all coefficients of $h(\mathbf{x}, \mathbf{y})$ are nonnegative. As we saw in Example 2.2, $h(\mathbf{x}, \mathbf{y})$ is stable if and only if the inequality (2) holds, which in this case amounts to

$$(ad - bc)^2 + a^2d^2 + b^2c^2 - 2ad(ad - bc) - 2bc(ad - bc) - 2adbc \leq 0,$$

or equivalently

$$-4bc(ad - bc) \leq 0.$$

Since $b \geq 0$, $c \geq 0$, and $ad - bc \geq 0$, the result follows. \square

3 Total positivity

A matrix $A \in \text{Mat}(n \times n)$ is *totally positive* if all of its minors are strictly positive. We denote the set of totally positive $n \times n$ matrices by $\text{Mat}_{>0}(n \times n)$, and we denote the set of totally nonnegative matrices by $\text{Mat}_{\geq 0}(n \times n)$. Lying between these is the set $\text{GL}_{\geq 0}(n) = \text{Mat}_{\geq 0}(n \times n) \cap \text{GL}(n)$ of invertible totally nonnegative matrices. Each of the sets $\text{Mat}_{\geq 0}(n \times n)$, $\text{Mat}_{>0}(n \times n)$ and $\text{GL}_{\geq 0}(n)$ is a multiplicative semigroup, i.e. closed under matrix multiplication. We have containments

$$\text{Mat}_{>0}(n \times n) \subset \text{GL}_{\geq 0}(n) \subset \text{Mat}_{\geq 0}(n \times n)$$

and $\text{Mat}_{\geq 0}(n \times n)$ is the closure of all of these sets [20].

The Loewner–Whitney theorem [12] describes the generators of $\text{GL}_{\geq 0}(n)$. Let

$$D_i(t) := \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & & \vdots & \vdots \\ 0 & 0 & & t & & 0 & 0 \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 1 \end{pmatrix} \quad E_i(t) := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & & & \vdots & \vdots \\ 0 & 0 & & 1 & t & & 0 & 0 \\ 0 & 0 & & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where in each case, t appears in row i ; let $F_i(t)$ be the transpose of $E_i(t)$. $\text{GL}_{\geq 0}(n)$ is the semigroup generated by all $D_i(t)$, $E_i(t)$, $F_i(t)$, $t > 0$. We use this description to prove Theorem 1.2.

Proof of Theorem 1.2. We begin with the implication (a) \Rightarrow (b). We will show that if $A \in \text{GL}_{\geq 0}(n)$ then $A_{\#}$ is a true stability preserver. Since $\text{Mat}_{\geq 0}(n \times n)$ is the closure of $\text{GL}_{\geq 0}(n)$, and the set of true stability preservers is closed, this implies the result for $A \in \text{Mat}_{\geq 0}(n \times n)$.

Since $\text{GL}_{\geq 0}(n)$ is a semigroup, and $(AB)_{\#} = A_{\#}B_{\#}$ for $A, B \in \text{Mat}(n \times n)$, it suffices to prove this in the case where A is a generator for $\text{GL}_{\geq 0}(n \times n)$, i.e. one of $D_i(t)$, $E_i(t)$, $F_i(t)$, $t > 0$. In each case, we can write

$$A = \begin{pmatrix} I_k & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & I_{n-k-2} \end{pmatrix}$$

where $0 \leq k \leq n-2$, and $Q = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is some totally nonnegative 2×2 matrix. Observe that

$$A_{\#}\mathbf{x}^I = \begin{cases} \mathbf{x}^J & \text{if } k+1 \notin I, k+2 \notin I \\ (ax_{k+1} + bx_{k+2})\mathbf{x}^J & \text{if } k+1 \in I, k+2 \notin I \\ (cx_{k+1} + dx_{k+2})\mathbf{x}^J & \text{if } k+1 \notin I, k+2 \in I \\ (ad - bc)x_{k+1}x_{k+2}\mathbf{x}^J & \text{if } k+1 \in I, k+2 \in I. \end{cases} \quad (4)$$

where $J = I \setminus \{k+1, k+2\}$. Comparing (4) with (3), we see that $A_{\#}$ is the unique $\mathbb{C}[x_1, \dots, x_k, x_{k+3}, \dots, x_n]$ -linear extension of $Q_{\#} : \mathbb{C}^{\text{MA}}[x_{k+1}, x_{k+2}] \rightarrow \mathbb{C}^{\text{MA}}[x_{k+1}, x_{k+2}]$. By Lemma 2.6, $Q_{\#}$ is a true stability preserver, and therefore by Proposition 2.5 so is $A_{\#}$.

For the implication (b) \Rightarrow (a), suppose that $A_{\#}$ is a stability preserver. If A is the zero matrix, then A is certainly totally nonnegative. Otherwise, $A_{\#}$ has rank at least 2, so by Theorem 2.4, it must be a true stability preserver, i.e.

$$h(\mathbf{x}, \mathbf{y}) = A_{\#} \left(\prod_{i=1}^n (x_i + y_i) \right)$$

is stable. Since $A_{\#}$ preserves degree, $h(\mathbf{x}, \mathbf{y})$ is homogeneous of degree n , and since $A_{\#}$ acts trivially on constants, the coefficient of $\mathbf{y}^{[n]}$ in $h(\mathbf{x}, \mathbf{y})$ is 1. Therefore by the phase theorem all coefficients of $h(\mathbf{x}, \mathbf{y})$ must be nonnegative. More generally the coefficient of $\mathbf{x}^I \mathbf{y}^J$ in $h(\mathbf{x}, \mathbf{y})$ is the minor of A corresponding to row set $[n] \setminus J$, and column set I . Since all minors of A are coefficients of $h(\mathbf{x}, \mathbf{y})$, we deduce that all minors of A are nonnegative. \square

The *totally positive part* of the Grassmannian $\text{Gr}(k, n)$, denoted $\text{Gr}_{>0}(k, n)$ is the set of $V \in \text{Gr}(k, n)$ such that all Plücker coordinates of V are strictly positive. Since totally

positive matrices are invertible, they act on the Grassmannian $\text{Gr}(k, n)$, and the totally positive part of the Grassmannian is an “orbit”. Specifically, let $V_0 \in \text{Gr}_{\geq 0}(k, n)$ be the column space of $M_0 = \begin{pmatrix} I_k \\ 0 \end{pmatrix}$. Then we have $\text{Gr}_{>0}(k, n) = \{AV_0 \mid A \in \text{Mat}_{>0}(n \times n)\}$, where AV_0 is defined to be the column space of the matrix AM_0 . The totally nonnegative part of the Grassmannian $\text{Gr}_{\geq 0}(k, n)$ does not have such a straightforward relationship to $\text{Mat}_{\geq 0}(n \times n)$, but is the closure of $\text{Gr}_{>0}(k, n)$. These facts are essentially Lusztig’s definitions of $\text{Gr}_{>0}(k, n)$ and $\text{Gr}_{\geq 0}(k, n)$ [13].

Proof of Theorem 1.1. As already noted in the introduction, if $f(\mathbf{x})$ is a stable polynomial representing $V \in \text{Gr}(k, n)$, then by the phase theorem, $V \in \text{Gr}_{\geq 0}(k, n)$. It remains to prove that if $f(\mathbf{x})$ represents a point $V \in \text{Gr}_{\geq 0}(k, n)$, then $f(\mathbf{x})$ is stable.

Since $\text{Gr}_{\geq 0}(k, n)$ is the closure of $\text{Gr}_{>0}(k, n)$, and since the set of multiaffine stable polynomials is closed, it suffices to prove the theorem when $V \in \text{Gr}_{>0}(k, n)$. If this is the case, there exists a totally positive matrix $A \in \text{Mat}_{>0}(n \times n)$ such that $V = AV_0$. Note that the monomial $\mathbf{x}^{[k]}$ represents $V_0 \in \text{Gr}_{\geq 0}(k, n)$. Since the action of $A_{\#}$ on multiaffine polynomials is defined via an isomorphism with the exterior algebra, we have that $f(\mathbf{x}) = A_{\#}\mathbf{x}^{[k]}$. By Theorem 1.2, $A_{\#}$ is a stability preserver, and $\mathbf{x}^{[k]}$ is stable, so $f(\mathbf{x})$ is stable. \square

4 Odds and ends

There is a second connection between Theorems 1.1 and 1.2. If $A \in \text{Mat}(n \times n)$, let $A^{\vee} \in \text{Mat}(n \times n)$ denote the matrix $A_{i,j}^{\vee} = (-1)^{n-j}A_{n+1-i,j}$. Let $V \in \text{Gr}(n, 2n)$ be the column space of the $2n \times n$ matrix $\begin{pmatrix} I_n \\ A^{\vee} \end{pmatrix}$. It is not hard to check the following facts:

- $V \in \text{Gr}_{\geq 0}(n, 2n)$ if and only if $A \in \text{Mat}_{\geq 0}(n \times n)$.
- $A_{\#}(\prod_{i=1}^n (x_i + y_i))$ represents V , with the variables ordered $y_1 < y_2 < \dots < y_n < x_n < \dots < x_2 < x_1$.

Thus we see that Theorem 1.1 implies Theorem 1.2, though not by reversing the argument in Section 3: $A_{\#}$ is a stability preserver iff $A_{\#}(\prod_{i=1}^n (x_i + y_i))$ is stable iff $V \in \text{Gr}_{\geq 0}(n, 2n)$ iff $A \in \text{Mat}_{\geq 0}(n \times n)$.

There is another class of stable polynomials comes that from the minors of a matrix. If $M \in \text{Mat}(n \times k)$, then the polynomial

$$\sum_{I \in \binom{[n]}{k}} |\det(M[I])|^2 \mathbf{x}^I \in \mathbb{C}^{\text{MA}}[\mathbf{x}] \quad (5)$$

is always stable [5, Theorem 8.1]. This raises the question: to what extent do these classes overlap?

The answer is not much. For dimensional reasons, a general polynomial of the form (5) does not represent a point of $\text{Gr}_{\geq 0}(k, n)$. On the other hand, with the exception of

a few small cases, a point of $\text{Gr}_{>0}(k, n)$ cannot be represented by a polynomial of the form (5). For ease of notation, we present the argument for $\text{Gr}(2, 6)$, though the same idea works for $k \geq 2, n - k \geq 4$.

Proposition 4.1. *No point of $\text{Gr}_{>0}(2, 6)$ is represented by a polynomial of the form (5).*

Proof. Suppose to the contrary that $\sum a_I \mathbf{x}^I$ represents a point of $\text{Gr}_{>0}(2, 6)$, and $a_I = |b_I|^2$ where $b_I = \det(M[I])$ for some matrix M . Then both $[a_I : I \in \binom{[6]}{2}]$ and $[b_I : I \in \binom{[6]}{2}]$ satisfy the Plücker relations. For $\text{Gr}(2, 6)$ these are:

$$\begin{aligned} a_{ik}a_{jl} &= a_{ij}a_{kl} + a_{il}a_{jk} \\ b_{ik}b_{jl} &= b_{ij}b_{kl} + b_{il}b_{jk} \end{aligned}$$

for $1 \leq i < j < k < l \leq 6$. Multiplying the second equation by its complex conjugate and using $a_I = |b_I|^2$, we find that $b_{ij}b_{kl}\overline{b_{il}b_{jk}}$ and $\overline{b_{ij}b_{kl}}b_{il}b_{jk}$ are pure imaginary. In particular,

$$b_{12}b_{35}\overline{b_{15}b_{23}} \quad \overline{b_{12}b_{36}}b_{16}b_{23} \quad \overline{b_{34}b_{56}}b_{36}b_{45} \quad \overline{b_{13}b_{45}}b_{15}b_{34} \quad b_{13}b_{56}\overline{b_{16}b_{35}}$$

are all pure imaginary. The product of these five pure imaginary numbers must be pure imaginary. But instead, their product is $a_{12}a_{13}a_{15}a_{16}a_{23}a_{34}a_{35}a_{36}a_{45}a_{56} > 0$. This is a contradiction. \square

A related result replaces the determinant of $M[I]$ with the permanent. If $M \in \text{Mat}(n \times k)$ is a matrix with nonnegative real entries then

$$\sum_{I \in \binom{[n]}{k}} \text{per}(M[I]) \mathbf{x}^I \in \mathbb{C}^{\text{MA}}[\mathbf{x}] \quad (6)$$

is a stable polynomial [5, Theorem 10.2]. It would be surprising if it were typically possible to represent a point of $\text{Gr}_{>0}(k, n)$ by a polynomial of the form (6). For example, it is not hard to show this impossible if $k = 2, n \geq 5$, but at present we do not have a general proof.

A multiaffine polynomial $f(\mathbf{x}) \in \mathbb{R}^{\text{MA}}[\mathbf{x}]$ is said to be a *Rayleigh polynomial* if $\Delta_{ij}f : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}$ is nonnegative for all $i, j \geq 0$. This is a relaxation of than the criterion for stability in Theorem 2.1: for the not-so-keenly observant, the Rayleigh condition only requires $\Delta_{ij}f \geq 0$ on nonnegative inputs, whereas stability requires $\Delta_{ij}f \geq 0$ on all real inputs. Thus real multiaffine stable polynomials are Rayleigh, but in general the converse is not true. Given $V \in \text{Gr}(k, n)$, let $\mathcal{B} \subset \binom{[n]}{k}$ be the set of indices of the nonzero Plücker coordinates of V . \mathcal{B} is the (set of bases of) a representable matroid, and if $V \in \text{Gr}_{\geq 0}(k, n)$, \mathcal{B} is called a positroid. Marcott has recently proved the following result.

Theorem 4.2 (Marcott [8]). *If \mathcal{B} is a positroid, then $B(\mathbf{x}) = \sum_{I \in \mathcal{B}} x^I$ is a Rayleigh polynomial.*

This is much like the harder direction of Theorem 1.1, except that the coefficients in the polynomial have been stripped away. The converse of Theorem 4.2 is not true: if \mathcal{B} is matroid that is not a positroid, then $B(\mathbf{x})$ may or may not be Rayleigh; there is no known classification of Rayleigh matroids. It is also not presently known whether, for positroids, $B(\mathbf{x})$ is a stable polynomial.

We mention two applications of the ideas developed in this paper. A linear endomorphism $\delta : \mathbb{C}^{\text{MA}}[\mathbf{x}] \rightarrow \mathbb{C}^{\text{MA}}[\mathbf{x}]$ is an *infinitesimal stability preserver* if $\exp(t\delta) : \mathbb{C}^{\text{MA}}[\mathbf{x}] \rightarrow \mathbb{C}^{\text{MA}}[\mathbf{x}]$ is a stability preserver for all $t \geq 0$. The set of all infinitesimal stability preservers is a closed convex cone in the space of all operators on $\mathbb{C}^{\text{MA}}[\mathbf{x}]$. This can be seen as follows: if α and β are infinitesimal stability preservers, then for $t \geq 0$, $\exp(t(\alpha + \beta)) = \lim_{m \rightarrow \infty} \left(\exp(\frac{t}{m}\alpha) \exp(\frac{t}{m}\beta) \right)^m$ is a stability preserver, and hence $\alpha + \beta$ is an infinitesimal stability preserver.

Our first application is an example of a non-trivial family of infinitesimal stability preservers. Let $Z \in \text{Mat}(n \times n)$ be a matrix with real diagonal entries, and nonnegative off-diagonal entries. Define $\delta_Z : \mathbb{C}^{\text{MA}}[\mathbf{x}] \rightarrow \mathbb{C}^{\text{MA}}[\mathbf{x}]$ by

$$\delta_Z x^J = \sum_{j \in J} \left(Z_{jj} + \sum_{i \in [n] \setminus J} Z_{ij} \frac{x_i}{x_j} \right) x^J$$

for all $J \subset [n]$, and extending linearly.

Proposition 4.3. *If Z is tridiagonal (i.e. $Z_{ij} = 0$ for $|i - j| > 1$), then for all $t \geq 0$, $\exp(tZ)$ is totally nonnegative, and $\exp(t\delta_Z) = \exp(tZ)_\#$.*

Proof. The fact that $\exp(tZ)$ is totally nonnegative follows from the Loewner–Whitney theorem, one formulation of which is that matrices of this form infinitesimally generate $\text{GL}_{\geq 0}(n)$. To see that $\exp(t\delta_Z) = \exp(tZ)_\#$, we need to verify that $\delta_Z = \frac{\partial}{\partial t} \exp(tZ)_\# \Big|_{t=0}$. But since $Z \mapsto \delta_Z$, and $Z \mapsto \frac{\partial}{\partial t} \exp(tZ)_\# \Big|_{t=0}$ are both linear maps, it suffices to check this when Z has a single nonzero entry; this is straightforward. \square

Proposition 4.4. *For any $Z \in \text{Mat}(n \times n)$ with real diagonal entries, and nonnegative off-diagonal entries, δ_Z is an infinitesimal stability preserver.*

Proof. First suppose Z is tridiagonal. In this case, by Proposition 4.3 and Theorem 1.2, we have that $\exp(t\delta_Z) = \exp(tZ)_\#$ is a stability preserver for all $t \geq 0$; hence δ_Z is an infinitesimal stability preserver.

Next suppose that $Z = Q_1 Z_1 Q_1^{-1}$ for some permutation matrix Q_1 and some tridiagonal matrix Z_1 . Since the definition of δ_Z is symmetric in variables x_1, \dots, x_n , it is clear that δ_Z is an infinitesimal stability preserver in this case too.

Finally observe that a general Z can be written as

$$Z = \sum_{i=1}^s Q_i Z_i Q_i^{-1}$$

where each Z_i is a real tridiagonal matrix with nonnegative off-diagonal entries, and Q_i is a permutation matrix. Since the map $Z \mapsto \delta_Z$ is linear, we see that δ_Z is a sum of infinitesimal stability preservers, and the result follows. \square

Remark 4.5. The proof of Proposition 4.4 is fundamentally the same as the proof of [3, Proposition 5.1], which also establishes a family of infinitesimal stability preservers. The two families are superficially similar but neither is a special case of the other. Concretely, the operators in [3] are given by $x^J \mapsto \sum_{j \in J} \sum_{i \in [n] \setminus J} Z_{ij} \left(\frac{x_i}{x_j} - 1 \right) x^J$ for a real symmetric matrix Z ; the exponentials of the operators in this family are doubly stochastic, and have the physical interpretation as generators for a symmetric exclusion process on n sites. Proposition 4.4 seems to be about the best one can do to mimic this construction for asymmetrical matrices.

As a second application, we prove a slightly more general version of the phase theorem.

Theorem 4.6. *Let $f(\mathbf{x}) \in \mathbb{C}[\mathbf{x}]$ be a stable polynomial. If $f(\mathbf{x})$ has no terms of degree k , $k \in \mathbb{Z}$, then there exists a nonzero scalar $\alpha \in \mathbb{C}^\times$ such that of all terms of degree $k+1$ in $\alpha f(\mathbf{x})$ and all terms of degree $k-1$ in $-\alpha f(\mathbf{x})$ have nonnegative real coefficients.*

Remark 4.7. There cannot be large gaps in the degrees of a stable polynomial: if $f(\mathbf{x})$ is stable and has no terms of degree k , then either $f(\mathbf{x})$ has terms of *both* degree $k+1$ and $k-1$, or $k > \max \deg f(\mathbf{x})$, or $k < \min \deg f(\mathbf{x})$. This can be deduced from the corresponding fact for single variable polynomials, or from an argument similar to the one presented below. It follows that Theorem 4.6 also implies the stronger version of the phase theorem in [5, Theorem 6.2].

The *support* of a polynomial $f(\mathbf{x})$ is the set of monomials in $\mathbb{C}[\mathbf{x}]$ that appear in $f(\mathbf{x})$ with a nonzero coefficient. Define $\|f(\mathbf{x})\|$ to be the maximum of the absolute values of the coefficients of $f(\mathbf{x})$. For example, if $f(\mathbf{x}) = 4x_1x_2^2 - x_1^3$, then the support of $f(\mathbf{x})$ is $\{x_1x_2^2, x_1^3\}$, and $\|f(\mathbf{x})\| = 4$.

Lemma 4.8. *Let $f(\mathbf{x}) \in \mathbb{C}^{\text{MA}}[\mathbf{x}]$ be a multiaffine stable polynomial. For every $\varepsilon > 0$, there exists a polynomial $f_\varepsilon(\mathbf{x}) \in \mathbb{C}^{\text{MA}}[\mathbf{x}]$ such that*

- (i) $f_\varepsilon(\mathbf{x})$ is stable;
- (ii) $\|f(\mathbf{x}) - f_\varepsilon(\mathbf{x})\| < \varepsilon$;
- (iii) for all $k \in \mathbb{Z}$, if $f(\mathbf{x})$ has no terms of degree k then $f_\varepsilon(\mathbf{x})$ has no terms of degree k ; and
- (iv) if $f(\mathbf{x})$ has a term of degree k , then the support of $f_\varepsilon(\mathbf{x})$ contains all multiaffine monomials of degree k .

Proof. Take $f_\varepsilon(\mathbf{x}) \in \mathbb{C}^{\text{MA}}[\mathbf{x}]$ such that (i)–(iii) above are satisfied, and subject to these conditions $f_\varepsilon(\mathbf{x})$ has maximal support. We claim that $f_\varepsilon(\mathbf{x})$ must also satisfy property (iv). If not, then we can find a matrix $A = E_i(t)$ or $F_i(t)$ such that for all but finitely many $t \in \mathbb{R}$, $A_\# f_\varepsilon(\mathbf{x})$ has strictly larger support than $f_\varepsilon(\mathbf{x})$. By taking $t > 0$ sufficiently small, we can achieve $\|f_\varepsilon(\mathbf{x}) - A_\# f_\varepsilon(\mathbf{x})\| < \varepsilon - \|f(\mathbf{x}) - f_\varepsilon(\mathbf{x})\|$. Thus, $\|f(\mathbf{x}) - A_\# f_\varepsilon(\mathbf{x})\| < \varepsilon$, i.e. $A_\# f_\varepsilon(\mathbf{x})$ satisfies (ii). By Theorem 1.2, and $A_\# f_\varepsilon(\mathbf{x})$ satisfies (i), and since $A_\#$ preserves degree, $A_\# f_\varepsilon(\mathbf{x})$ satisfies (iii). Thus we have a contradiction in the choice of $f_\varepsilon(\mathbf{x})$. \square

Proof of Theorem 4.6. By the Grace–Walsh–Szegő coincidence theorem [7, 17, 19] it suffices to prove this in the case where $f(\mathbf{x}) \in \mathbb{C}^{\text{MA}}[\mathbf{x}]$ is a multiaffine polynomial.

Consider $f_\varepsilon(\mathbf{x})$, $\varepsilon > 0$. For any $I \subset [n]$, and any $i, j \in [n] \setminus I$ we can write

$$f_\varepsilon(\mathbf{x}) = \mathbf{x}^I(a + bx_i + cx_j + dx_ix_j) + \dots$$

where the \dots indicates all terms that are not of this form. It is straightforward (using Theorem 2.4 or elementary arguments) to check that the linear map $\phi : \mathbb{C}[\mathbf{x}] \rightarrow \mathbb{C}[\mathbf{x}]$ defined by

$$\phi(x^J) = \begin{cases} x^{J \setminus I} & \text{if } I \subset J \subset I \cup \{i, j\} \\ 0 & \text{otherwise} \end{cases}$$

is a stability preserver. Thus $\phi(f_\varepsilon(\mathbf{x})) = a + bx_i + cx_j + dx_ix_j$ is stable and multiaffine in two variables. If $|I| = k$, then $a = 0$ from which it is easy to show that b and c have same phase. By property (iv) of $f_\varepsilon(\mathbf{x})$, $b \neq 0$ iff $c \neq 0$, which implies that the “same phase” relation is transitive. Thus by considering all I with $|I| = k$, we see that all terms of degree $k + 1$ in $f_\varepsilon(\mathbf{x})$ have the same phase. Similarly $|I| = k - 2$, then $d = 0$ and we have the same result for terms of degree $k - 1$. If $|I| = k - 1$ then $b = c = 0$, and we deduce that d and $-a$ have the same phase. This shows that the result is true for the polynomial $f_\varepsilon(\mathbf{x})$. The theorem now follows, since $f(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0} f_\varepsilon(\mathbf{x})$. \square

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